OPTIMAL REDUCED-RANK TIME-FREQUENCY/TIME-SCALE QUADRATIC DETECTORS

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ABSTRACT

Optimal detectors based on time-frequency/time-scale representations (TFRs/TSRs) admit a representation in terms of a bank of spectrograms/scalograms that yields a large class of detectors. These range from the conventional matched filter to the more complex higher-rank detectors offering a superior performance in a wider variety of detection situations. In this paper, we optimize this complexity versus performance tradeoff by characterizing TFR/TSR detectors that optimize performance (based on the deflection criterion) for any given fixed rank. We also characterize the gain in performance as a function of increasing complexity thereby facilitating a judicious tradeoff. Our experience with real data shows that, in many cases, relatively low-rank optimal detectors can provide most of the gain in performance relative to matched-filter processors.

1. INTRODUCTION

Spurred by the need for detection in nonstationary environments, recently there has been substantial interest in the use of time-frequency and time-scale representations (TFRs and TSRs) for detection. For example, TFR/TSR-based detection has been explored in mechanical diagnostics applications such as machine monitoring [1] and engine knock detection [2]. More recently, a theoretical framework for optimal time-frequency/time-scale detection has been developed that puts such primarily heuristic approaches on a firm footing [3]; it characterizes the scenarios in which bilinear time-frequency/time-scale detectors are optimal and derives the corresponding TFR/TSR-based processors.

The short-time Fourier transform (STFT), which can be interpreted as a narrowband cross-ambiguity function (AF), is one of the simplest TFRs and has long been used in radar/sonar detection: It efficiently implements the matched-filter detector corresponding to unknown time (range) and frequency (Doppler) shifts.[1] Similarly, the wideband AF is essentially the wavelet transform (WT). Spectrograms/scalograms (squared magnitudes of STFT/WT) are used in noncoherent situations[2], and provide simple and efficient detector structures.

The quadratic TFR/TSR-based detectors characterized in [3], by virtue of their more general structure, facilitate optimal detection in a broader variety of scenarios: Essentially any second-order random signal with unknown or random time-frequency/time-scale parameters, embedded in arbitrary Gaussian noise, can be optimally detected. Interestingly, these more complex detectors can be realized as a weighted sum of a bank of spectrogram/scalograms. This fact is used in [3] to derive optimal detector structures based on partial signal information. In this paper, we exploit this subspace-based formulation to optimize the *complexity versus performance* tradeoff between the simple spectrogram/scalogram-based matched filter detectors applicable in specific scenarios, and the more complex quadratic TFR/TSR detectors suitable in a wider variety of situations.

More specifically, given a fixed level of complexity as determined by the number, N_r, of spectrograms/scalograms in the bank, we determine the optimal windows, and the corresponding optimal combining weights, that characterize the N_r processors. We use "deflection" as the optimality criterion,[3] since it is a measure of signal-to-noise ratio (SNR) and hence reflects detection performance [4]. The optimal combining weights satisfy a monotonicity property which results in monotonically decreasing gain in performance for each additional spectrogram/scalogram in the bank. Thus, our results facilitate a judicious choice for the size of the bank (and hence complexity), and determine the optimal detector for that size.

In the next section, we provide a brief description of the quadratic TFR/TSR detectors, and their subspace-based formulation, as derived in [3]. Section 3 characterizes the optimal reduced-rank TFR/TSR detectors and describes some of their useful properties. Some concluding remarks are presented in Section 4.

2. OPTIMAL QUADRATIC TFR/TSR-BASED DETECTORS

It is shown in [3] that TFR/TSR-based detectors are optimal for composite hypothesis testing scenarios of the form

$$H_0 \quad : \quad x(t) = n(t)$$

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[1] Derived from the generalized likelihood ratio test (GLRT) for detecting an essentially deterministic signal with unknown time-frequency shifts in white Gaussian noise.

[2] Unknown or random signal amplitude and phase

[3] Using the probability of error, or the Neyman-Pearson criterion [4], as the performance measure is analytically intractable.

$$H_1 \quad : \quad x(t) = s(t;\alpha,\beta) + n(t)\ ,\ (\alpha,\beta) \in \mathbb{R}^2 \qquad (1)$$

where $t \in T$, the observation interval, x is the observed signal, $s(\alpha,\beta)$ is an arbitrary zero-mean, second-order (possibly Gaussian) stochastic signal with unknown or random parameters (α,β), and n is arbitrary zero-mean Gaussian noise independent of $s(\alpha,\beta)$. The noise is characterized by the correlation function $R_n(t_1,t_2) \equiv E[n(t_1)n^*(t_2)]$, and the signal is characterized (up to second-order statistics) by the correlation function $R_s^{(\alpha,\beta)}$. Based on the observed signal, x, it is to be decided whether the signal $s(\alpha,\beta)$ is present (H_1) or not (H_0). The optimal decision is made by comparing a real-valued test statistic, $L(x)$, to a threshold. The optimal test statistic, in all the scenarios discussed in [3], is a quadratic function of the observations. As explained next, the parameters (α,β) correspond to time-frequency or time-scale for TFR or TSR detectors, respectively.

2.1. Signal Models

The appropriate signal models in (1) for which quadratic TFRs and TSRs form canonical detectors are characterized by [3]

$$\begin{aligned}
\text{TFRs} : (\alpha,\beta) &\longleftrightarrow (\tau,\nu) \in \mathbb{R}^2 \\
R_s^{(\tau,\nu)}(t_1,t_2) &= R_{TF}(t_1-\tau, t_2-\tau)e^{j2\pi\nu(t_1-t_2)} \qquad (2) \\
\text{TSRs} : (\alpha,\beta) &\longleftrightarrow (\tau,c) \in \mathbb{R} \times (0,\infty) \\
R_s^{(\tau,c)}(t_1,t_2) &= cR_{TS}(c(t_1-\tau), c(t_2-\tau)) \qquad (3)
\end{aligned}$$

for some correlation functions R_{TF} and R_{TS}, where $(t_1,t_2) \in T \times T$. The correlation functions R_{TF} and R_{TS} are fundamental to the structure of TFR/TSR detectors [3].

2.2. Characterization of TSR/TSR Detectors

For the above signal models in (1), the corresponding optimal TFR/TSR-based detectors[4] for a variety of scenarios are of the form [3]

$$\begin{aligned}
L_{TF}(x) &= \max_{(\tau,\nu)} [(Py)(\tau,\nu;\Phi) + F_{TF}(\tau,\nu)]\ , \qquad (4) \\
L_{TS}(x) &= \max_{(\tau,c)} [(Cy)(\tau,1/c;\Pi) + F_{TS}(\tau,c)]\ , \qquad (5)
\end{aligned}$$

for some functions F_{TF} and F_{TS} independent of x, where $P(\Phi)$ is a bilinear TFR from Cohen's class characterized by the kernel Φ [5], and $C(\Pi)$ is a bilinear TSR from the affine class characterized by the kernel Π [6]. In (4) and (5), $y = \mathbf{R}_n^{-1}x$, where $\mathbf{R}_n$ is the operator defined by the noise correlation function as

$$(\mathbf{R}_n x)(t) \equiv \int R_n(t,u)x(u)du\ , \qquad (6)$$

and we assume that $\mathbf{R}_n^{-1}$ exists.[5] Moreover, the characterizing kernels Φ and Π are completely characterized by $\{R_n, R_{TF}\}$ and $\{R_n, R_{TS}\}$, respectively.

[4]Based on a GLRT using maximum likelihood estimates of the parameters in the case of unknown parameters, and maximum *a posteriori* probability estimates for random parameters [3].

[5]The presence of a white noise component guarantees the existence of $\mathbf{R}_n^{-1}$.

2.3. Subspace-Based Formulation

The STFT with respect to a window h is defined as [5]

$$STFT_x(t,f;h) \equiv \int x(u)h^*(u-t)e^{-j2\pi fu}du\ , \qquad (7)$$

and the WT is defined as [7]

$$WT_x(t,a;g) \equiv \frac{1}{\sqrt{a}} \int x(u)g^*\left(\frac{u-t}{a}\right)du\ , \qquad (8)$$

where g is called the analysis or mother wavelet. The TFRs/TSRs in the detectors (4)/(5) can equivalently be expressed in terms of STFTs and WTs, respectively, as [3]

$$\begin{aligned}
L_{TF}^{(\tau,\nu)}(x) &\equiv (Py)(\tau,\nu;\Phi) \\
&= \sum_{k=1}^{N_{TF}} a_k |STFT_y(\tau,\nu;u_k)|^2 \qquad (9) \\
L_{TS}^{(\tau,c)}(x) &\equiv (Cy)(\tau,1/c;\Pi) \\
&= \sum_{k=1}^{N_{TS}} b_k |WT_y(\tau,1/c;v_k)|^2 \qquad (10)
\end{aligned}$$

where $y = \mathbf{R}_n^{-1}x$, the u_k's and the v_k's are the eigenfunctions of R_{TF} and R_{TS}, respectively, the a_k's and the b_k's are positive coefficients that depend on the corresponding eigenvalues, and N_{TF} and N_{TS} are the ranks of R_{TF} and R_{TS}.[6] This yields a subspace-based formulation of the TFR/TSR detectors: The detection of the random signal with unknown or random time-frequency/time-scale shifts is accomplished via a weighted combination of the squared-magnitudes of the projections onto the subspaces spanned by the signal eigenfunctions. Note that if the signal correlation functions R_{TF}/R_{TS} are rank-1, the above expansions reduce to the conventional matched-filter detectors. This representation of the detectors in terms of a bank of spectrograms/scalograms (rank-1 detectors) is exploited in the next section to derive optimal reduced-rank TFR/TSR detectors.

3. OPTIMAL REDUCED-RANK TFR/TSR DETECTORS

The number N_{TF}/N_{TS} of spectrograms/scalograms in the expansions (9)/(10) can be arbitrarily large depending on the complexity of the random signal to be detected. However, our experience with real data has indicated that in many cases the effective rank of the signal correlation function is relatively low. Thus, reduced-rank detectors, with substantially smaller number of spectrograms/scalograms in the bank, may be employed without any significant degradation in performance. In this section, we derive such optimal TFR/TSR detectors which yield the best performance for a given level of complexity (number of rank-1 spectrograms/scalograms).

[6]Rank is the number of nonzero eigenvalues in the eigenexpansion of the operators defined by these correlation functions as in (6).

Defining the unitary time-frequency shift operator as $(\mathbf{U}_{(\tau,\nu)}x)(t) \equiv e^{j2\pi\nu t}s(t-\tau)$ and the time-scale shift operator as $(\mathbf{U}_{(\tau,c)}x)(t) \equiv \sqrt{c}x(c(t-\tau))$, the TFR/TSR detectors (9) and (10) have the general quadratic form [3]

$$L_{TF}^{(\tau,\nu)}(x) = \left\langle \mathbf{Q}_{TF}\mathbf{U}_{(\tau,\nu)}^{-1}\mathbf{R}_n^{-1}x, \mathbf{U}_{(\tau,\nu)}^{-1}\mathbf{R}_n^{-1}x\right\rangle = \sum_{k=1}^{N_r}\lambda_k\left|\left\langle \mathbf{U}_{(\tau,\nu)}p_k, \mathbf{R}_n^{-1}x\right\rangle\right|^2 \quad (11)$$

$$L_{TS}^{(\tau,c)}(x) = \left\langle \mathbf{Q}_{TS}\mathbf{U}_{(\tau,c)}^{-1}\mathbf{R}_n^{-1}x, \mathbf{U}_{(\tau,c)}^{-1}\mathbf{R}_n^{-1}x\right\rangle = \sum_{k=1}^{N_r}\mu_k\left|\left\langle \mathbf{U}_{(\tau,c)}q_k, \mathbf{R}_n^{-1}x\right\rangle\right|^2 \quad (12)$$

for some positive definite operators $\mathbf{Q}_{TF}$ and $\mathbf{Q}_{TS}$, where the expansions are in terms of the eigenfunctions (p_k, q_k) and eigenvalues (λ_k, μ_k) of the operators.

Given the knowledge of signal and noise correlation functions (R_{TF}, R_{TS} and R_n), we are interested in finding rank-N_r ($N_r \leq N_{TF}, N_{TS}$) detectors that yield the best performance with respect to the deflection criterion (defined shortly) at the correct value of the parameters. Mathematically, this is equivalent to finding the best rank-N_r quadratic detector for the simple hypothesis testing problem

$$\begin{aligned} H_0 &: x(t) = n(t) \\ H_1 &: x(t) = s(t) + n(t) , \end{aligned} \quad (13)$$

where s has correlation function R_s, and n is Gaussian with correlation function R_n. In the context of composite hypothesis testing problem (1), $R_s = R_{TF}$ or R_{TS}, and such a reduced-rank detector maximizes deflection at the correct value of the parameters (α, β) (more will be said about this in Section 3.2).

3.1. Optimal Reduced-Rank Quadratic Detectors

A quadratic detector is completely characterized as

$$L_{\mathbf{Q}}(x) = \langle \mathbf{Q}x, x\rangle , \quad (14)$$

for some linear (Hermitian) operator $\mathbf{Q}$. For a quadratic detector $L_{\mathbf{Q}}$, the deflection is defined as [4, 3]

$$H(\mathbf{Q}) \equiv \frac{\left(E_1\left[L_{\mathbf{Q}}(x)\right] - E_0\left[L_{\mathbf{Q}}(x)\right]\right)^2}{\mathrm{Var}_0\left[L_{\mathbf{Q}}(x)\right]} \quad (15)$$

where E_i denotes the expectation under the hypothesis i, and Var_0 denotes the variance under H_0. Deflection is a measure of SNR, and the deflection-optimal detector[7], $\mathbf{Q}_D$, for the detection problem (13), is given by [4, 3]

$$\mathbf{Q}_D = \mathbf{R}_n^{-1}\mathbf{R}_s\mathbf{R}_n^{-1} . \quad (16)$$

Now, consider a rank-N_r detector of the form

$$\langle \mathbf{Q}_{N_r}x, x\rangle = \sum_{k=1}^{N_r}\gamma_k|\langle w_k, x\rangle|^2 \quad (17)$$

[7]The detector that maximizes deflection.

where $\gamma_k \in \mathbb{R}$ and the w_k's are linearly independent but not necessarily orthogonal to each other. The w_k's correspond to the "window functions" for the various spectrograms/scalograms in the TFR/TSR-detector expansions (9)/(10). The following result characterizes the best rank-N_r detector with respect to deflection.[8]

Theorem. The N_r optimal window functions w_k's in (17), which maximize deflection for the hypothesis testing problem (13), are

$$w_k^{opt} = \mathbf{R}_n^{-1/2}z_k\ ,\ \ k = 1, 2\cdots N_r, \quad (18)$$

where the z_k's are the N_r dominant eigenfunctions, corresponding to the N_r largest eigenvalues $\delta_1 \geq \delta_2 \cdots \geq \delta_{N_r}$, of the nonnegative definite SNR matrix [9][9]

$$\mathbf{S} \equiv \mathbf{R}_n^{-1/2}\mathbf{R}_s\mathbf{R}_n^{-1/2} . \quad (19)$$

The corresponding optimal weights γ_k's in (17) are proportional to the eigenvalues; that is,

$$\gamma_k^{opt} = a\delta_k\ ,\ \ k = 1, 2\cdots N_r, \quad (20)$$

for some $a > 0$, and the deflection of the corresponding optimal rank-N_r detector is given by

$$H(\mathbf{Q}_{N_r}^{opt}) = \sum_{k=1}^{N_r}\delta_k^2 . \quad (21)$$

Proof (sketch). It can be shown that, within a constant factor, the deflection $H(\mathbf{Q})$ for the hypotheses (13) is given by [4][10]

$$H(\mathbf{Q}) = \frac{\mathrm{Trace}^2(\mathbf{Q}\mathbf{R}_s)}{\mathrm{Trace}(\mathbf{Q}\mathbf{R}_n\mathbf{Q}\mathbf{R}_n)} . \quad (22)$$

Making the change of variables $\tilde{\mathbf{Q}} = \mathbf{R}_n^{1/2}\mathbf{Q}\mathbf{R}_n^{1/2}$ and representing $\tilde{\mathbf{Q}}$ in terms of its eigenexpansion, $\tilde{Q}(t_1,t_2) = \sum_k c_k g_k(t_1)g_k^*(t_2)$, (22) reduces to

$$H(\mathbf{Q}) = \frac{\mathrm{Trace}^2(\tilde{\mathbf{Q}}\mathbf{S})}{\mathrm{Trace}(\tilde{\mathbf{Q}}\tilde{\mathbf{Q}})} = \frac{|\sum_k c_k d_k|^2}{\sum_k |c_k|^2} \leq \sum_k |d_k|^2 \quad (23)$$

where $d_k = \langle \mathbf{S}g_k, g_k\rangle$ and the last inequality follows from the Cauchy-Schwarz inequality, with equality holding if $c_k = ad_k$, $a > 0$, for all k. We note that $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ have the same rank since $\mathbf{R}_n$ is assumed to be invertible. It follows that in order to maximize $H(\mathbf{Q})$ for a rank-N_r $\mathbf{Q}$, the g_k's should be chosen to be the N_r dominant eigenfunctions, z_k's, of $\mathbf{S}$, and the c_k's to be proportional to the corresponding eigenvalues δ_k's. The relation between $\tilde{\mathbf{Q}}$ and $\mathbf{Q}$ then implies that the w_k's and the γ_k's in the expansion for $\mathbf{Q}$ are given by (18) and (20). Moreover, (21) follows from (23) since $d_k = \delta_k$ for the optimal choice of the g_k's □.

[8]A similar result is derived in [8] in the context of quadratic beamformers.

[9]In [9], in a related but different problem, divergence-optimal reduced-rank approximations to optimal detectors are considered.

[10]For both complex and real Gaussian noise.

Note that for $N_r = \text{rank}(\mathbf{S}) = \text{rank}(\mathbf{R}_s)$, $\mathbf{Q}_{N_r}^{opt} = \mathbf{Q}_D$. Moreover, from (21) we note that the gain in performance due to the addition of a new rank-1 processor in the bank is

$$\frac{H(\mathbf{Q}_{N_r+1}^{opt}) - H(\mathbf{Q}_{N_r}^{opt})}{H(\mathbf{Q}_{N_r}^{opt})} = \frac{\delta_{N_r+1}^2}{\sum_{k=1}^{N_r} \delta_k^2} , \tag{24}$$

which shows that the gain is monotonically decreasing as a function of increasing rank (since $\delta_k \geq \delta_{k+1}$ for all k).

3.2. Optimal Reduced-Rank TFR/TSR Detectors

From (11) and (12), we note that the underlying operators of the TFR/TSR detectors are of the form $\mathbf{R}_n^{-1}\mathbf{Q}^{(\alpha,\beta)}\mathbf{R}_n^{-1}$ where $(\alpha, \beta) = (\tau, \nu)$ or (τ, c). The TFR/TSR-based structure of such quadratic detectors crucially depends on the appropriate dependence of $\mathbf{Q}^{(\alpha,\beta)}$ on the parameters (see the signal models in Section 2.1) [3], and the fact that $\mathbf{R}_n^{-1}$ can be incorporated into preprocessing of the signal. Thus, to preserve the TFR/TSR structure, we effectively need optimal reduced-rank approximation to the operator $\mathbf{Q}^{(\alpha,\beta)}$. By choosing $R_s = R_{TF}$ or R_{TS} in (13), and finding the corresponding deflection-optimal $\hat{\mathbf{Q}}$ for detectors (14) of the form $\mathbf{Q} = \mathbf{R}_n^{-1}\hat{\mathbf{Q}}\mathbf{R}_n^{-1}$, we effectively design optimal reduced-rank TFR/TSR detectors that maximize deflection at the correct value of the parameters $(\tau, \nu)/(\tau, c)$. It can be easily shown that $\hat{\mathbf{Q}}_{N_r}^{opt} = \mathbf{R}_n \mathbf{Q}_{N_r}^{opt} \mathbf{R}_n$, where $\mathbf{Q}_{N_r}^{opt}$ is derived in the Theorem. Thus, the N_r optimal window functions for the spectrograms/scalograms in (9)/(10) can be determined by replacing R_s with R_{TF}/R_{TS} in the Theorem and using the relations

$$u_k^{opt} = \mathbf{R}_n w_k^{opt} = \mathbf{R}_n^{1/2} z_k \ , \quad (R_s = R_{TF}) \tag{25}$$

$$v_k^{opt} = \mathbf{R}_n w_k^{opt} = \mathbf{R}_n^{1/2} z_k \ , \quad (R_s = R_{TS}) \tag{26}$$

for $k = 1, 2 \cdots N_r$, in conjunction with the preprocessing $y = \mathbf{R}_n^{-1} x$. Moreover, the optimal combining weights a_k^{opt}/b_k^{opt} are the same as the γ_k^{opt}'s determined in the Theorem for $R_s = R_{TF}/R_{TS}$.

3.3. Discussion

We first note that the optimal reduced-rank TFR/TSR detectors are not determined by the dominant eigenvectors of the signal correlation functions R_{TF}/R_{TS}. Instead, the dominant eigenvectors of the SNR matrix $\mathbf{S}$, defined in (19), determine the optimal structure.[11] Moreover, the optimal windows w_k^{opt}'s (or the u_k^{opt}'s) are not orthogonal in general. However, they decorrelate the data under both hypotheses: If we define $x_k = \langle x, w_k^{opt} \rangle$, then $E_0[|x_k|^2] = 1$ and $E_1[|x_k|^2] = (1 + \delta_k)$ for all k, and $E[x_k x_l^*] = 0$ for $k \neq l$ under both hypotheses.

The expression (21) for the deflection of the optimal N_r-rank detector provides a direct method for choosing the smallest N_r for a prescribed loss in performance; the deflection of the full-rank detector is given by (21) by replacing N_r with the rank N_{TF}/N_{TS} of the underlying signal correlation function R_{TF}/R_{TS}. Moreover, (24) quantifies the relative gain in performance for each additional rank-1 processor in the optimal reduced-rank detector.

[11] It is worth noting that in TFR/TSR detectors, the preprocessing by $\mathbf{R}_n^{-1}$ is different from the usual prewhitening transformation $\mathbf{R}_n^{-1/2}$.

4. CONCLUSIONS

The subspace-based formulation of optimal quadratic TFR/TSR detectors yields a class of detectors governed by a complexity versus performance tradeoff: Rank-1 spectrograms/scalograms are simple and efficient, whereas the more complex higher-rank detectors can yield a superior performance in a wide variety of detection scenarios. The results of this paper optimize this tradeoff by characterizing TFR/TSR detectors of any given fixed rank that yield optimal performance with respect to the deflection criterion. Our formulation also yields an explicit expression for the performance gain achieved by the addition of a new spectrogram/scalogram in the bank, which may be used to determine the lowest complexity (rank) for a prescribed tolerance for degradation in performance. Experiments with real data indicate that the effective rank of typical signals is indeed relatively small, making the general quadratic TFR/TSR detectors viable in practice by virtue of the optimal reduced-rank structures presented in this paper.

Finally, we note that similar reduced-rank structures can be derived for optimal detectors based on generalized joint signal representations that may be useful in situations in which signal parameters other than time, frequency or scale are more appropriate [10].

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